A STUDY OF THE TRAJECTORIES OF A MISSILE AS A FUNCTION

OF INITIAL ALTITUDE, INITIAL HORIZONTAL VELOCITY AND

BALLISTIC COEFFICIENT

A THESIS

Presented to

The Faculty of the Graduate Division

by

Herbert A. Hutchinson

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Aeronautical Engineering

Georgia Institute of Technology

February, 1959

A STUDY OF THE TRAJECTORIES OF A MISSILE AS A FUNCTION OF INITIAL ALTITUDE, INITIAL HORIZONTAL VELOCITY AND BALLISTIC COEFFICIENT

Approved:

Arnold L. Ducoffe

Jakob Mandelker

Frank M. White, Jr.

Date Approved by Chairman:

3 March, 1959

"In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

"

ACKNOWLEDGMENTS

The author expresses his appreciation to Doctor Arnold L. Ducoffe for suggesting the subject and giving invaluable assistance through all phases of the study. Gratitude is also extended to Doctor Jakob Mandelker and Doctor Frank M. White, Jr., for their critical review of the topic and to Mr. Benjamin Ronn and the several members of the Engineering Experiment Station of the Georgia Institute of Technology for their assistance in computing the data necessary for the study.

Sincere appreciation is extended to the Sandia Corporation for the financial and technical aid which made this work possible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

English

a	acceleration, feet per second per second
c	local speed of sound, feet per second
C_D	aerodynamic drag coefficient based on missile frontal area
$\overline{C}_D$	ratio of instantaneous C_D to minimum C_D
D	aerodynamic drag, pounds
F	force, pounds
g	gravitational acceleration, 32.17 feet per second per second
K	ballistic coefficient of the missile, square feet per slug
m	mass of the missile, slugs
M	Mach number, ratio of missile velocity to local speed of sound
S	frontal area of missile, square feet
t	time, seconds
V	velocity of the missile, feet per second
W	weight of the missile, pounds
x	horizontal coordinate of missile location, feet
$\overline{x}$	horizontal coordinate of convergence point location, feet
y	altitude or vertical coordinate of missile location, feet
$\overline{y}$	deviation of convergence point from initial altitude, feet

Greek

θ	angle between the tangent to a trajectory at a point and a horizontal reference through the point, degrees
ρ	local density of atmosphere, slugs per cubic foot

Subscripts

min	minimum quantity of the function
o	condition at $t = 0$
x	horizontal component of the vector
y	vertical component of the vector

Superscripts

'	first derivative with respect to time, velocity
"	second derivative with respect to time, acceleration

Results of the study indicate that the coordinates of the computed trajectories at a certain instant of time, t, lie essentially on a straight line. Furthermore, the straight lines constructed for each time, t, on each family of trajectories were found to converge to essentially a point of common intersection. The convergence point was subsequently found to vary in location as a function of the altitude and velocity initial conditions. Graphical plots, second degree polynomials and nomographs are presented which describe the variation of the point location with initial conditions.

A straight line connecting the predicted convergence point with a point computed for the vacuum trajectory at a time, t, determines a constant time line. Results of the study indicate that, if at least one other trajectory is known in addition to the corresponding vacuum trajectory, linear interpolation along constant time lines may be utilized over small ranges of the ballistic coefficient for the prediction of other trajectories.

Further study of the effect which the ballistic coefficient has in determining the regression of the coordinates along constant time lines with regard to the vacuum trajectory may provide a more useful means for predicting trajectories over a large range of ballistic coefficients.

CHAPTER I

INTRODUCTION

The study of the trajectories of missiles has been, and promises to be, a classical problem in dynamics for many years. Recent advances in missile technology have indicated the need for methods by which the trajectories of missiles may be rapidly predicted. Investigations are currently being made as to the feasibility of transporting cargo between distant geographical locations by means of missiles. Certainly a missile of fixed external configuration but having varied internal mass will have a different trajectory for each different mass. Knowledge of the trajectory is essential for delivery of the cargo to the desired target area. The same analogy applies to military applications of missiles whereby the destructive power of a weapon-type missile is a direct function of its location in relation to the target.

Many analytical treatments for predicting trajectories have been presented in many texts, of which Reference 1 is an excellent example. Until recently, computational means for verifying the analytical results required a prohibitive amount of hand-computations which often were in error. Modern advances in computational techniques, namely analog and digital computer machinery, have eliminated this undesirable feature of trajectory computations and studies of trajectory data may now be efficiently conducted.

The purpose of this research is to study the trajectories of a missile with various initial horizontal velocities at different altitudes.

In order to determine the effect of varying missile weight and size, trajectories, which have been obtained for various ratios of weight to frontal area for the chosen range of velocities and altitudes will be studied.

CHAPTER II

ASSUMPTIONS AND EQUATIONS

As is necessary in the derivation of equations of motion, simplifying assumptions must be made as to the type of motion which is to be considered. This study will be conducted for a freely-falling missile which will have only two-dimensional trajectories. Initial conditions are given by the altitude, y_o, and the initial horizontal velocity, $\dot{x}_o$, with the initial vertical velocity component zero (i.e. $\dot{y}_o = 0$). The missile will be of a non-spinning type and will be considered to have an ideal tractability, that is the longitudinal axis of the missile will remain tangent to the trajectory. Assumption of ideal tractability reduces the consideration of the aerodynamic forces to that of drag force alone. The drag coefficient will be assumed to vary with Mach number as shown in Figure 1. Empirical relations describing the variation are as shown in the appendix.

Additional assumptions must be made as to the medium through which the missile will have its trajectory. For the purposes of this study, it will be assumed that there are no wind effects and that the gravitational field is homogeneous. The local speed of sound, used in the computation of Mach number, and the local atmospheric density will be assumed to vary with altitude. Empirical relations defining the variation of the speed of sound and density with altitude are as shown in the appendix and have been obtained from Reference 2. The coordinate system will be selected such that the earth may be considered to have a plane surface and be non-rotating with respect to the two-dimensional

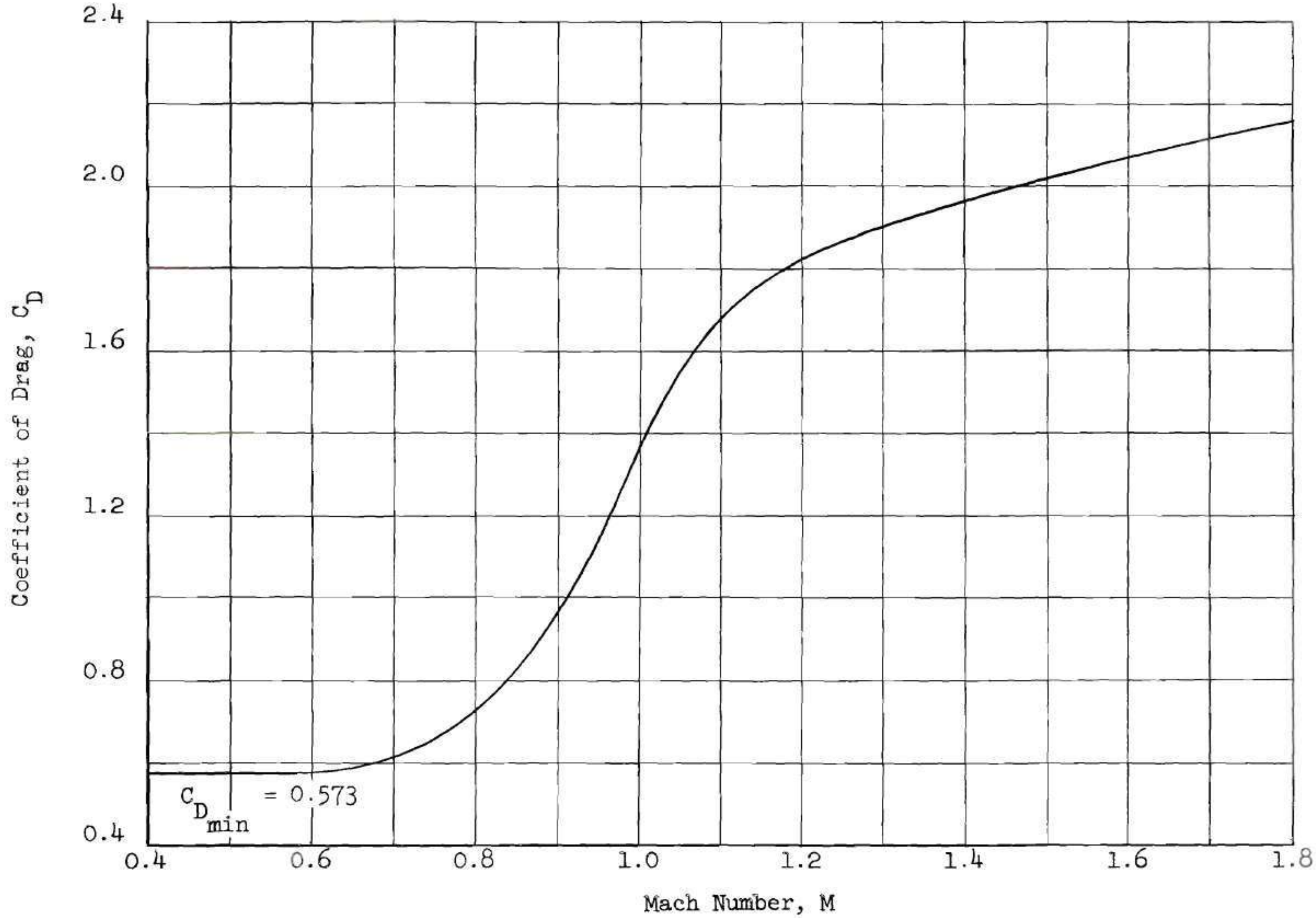

Figure 1. Aerodynamic Drag Characteristics of the Missile

coordinate system. Rotation and curvature of the earth should be accounted for in determining where the missile trajectory will intersect the surface of the earth; this study will be confined to that of the mutual relationship of the time-space coordinates of the trajectories.

The assumptions thus made allow the missile to be considered as a material point endowed with the physical characteristics of the missile. At any instant, the only forces acting upon the missile-point will be the weight of the missile and the aerodynamic drag force. Figure 2 is an illustration of this concept of the dynamics of the missile. The equations of motion may now be derived.

Utilization of Newton's second law of motion:

$$F = m\,a \tag{1}$$

enables the summation of the forces acting upon the missile at any time, t, as illustrated in Figure 2, to be written as:

$$\Sigma F_x = m\,x'' = -\,D\cos\theta \tag{2}$$

and

$$\Sigma F_y = m\,y'' = D\sin\theta - m\,g \tag{3}$$

where m is the mass of the missile, x'' and y'' are the horizontal and vertical components of the acceleration respectively, D is the drag force, and θ is the angle between the tangent to the trajectory at a point and a horizontal reference through the point.

Equations (2) and (3) may be simplified to:

$$x'' = -\,(D/m)\cos\theta \tag{4}$$

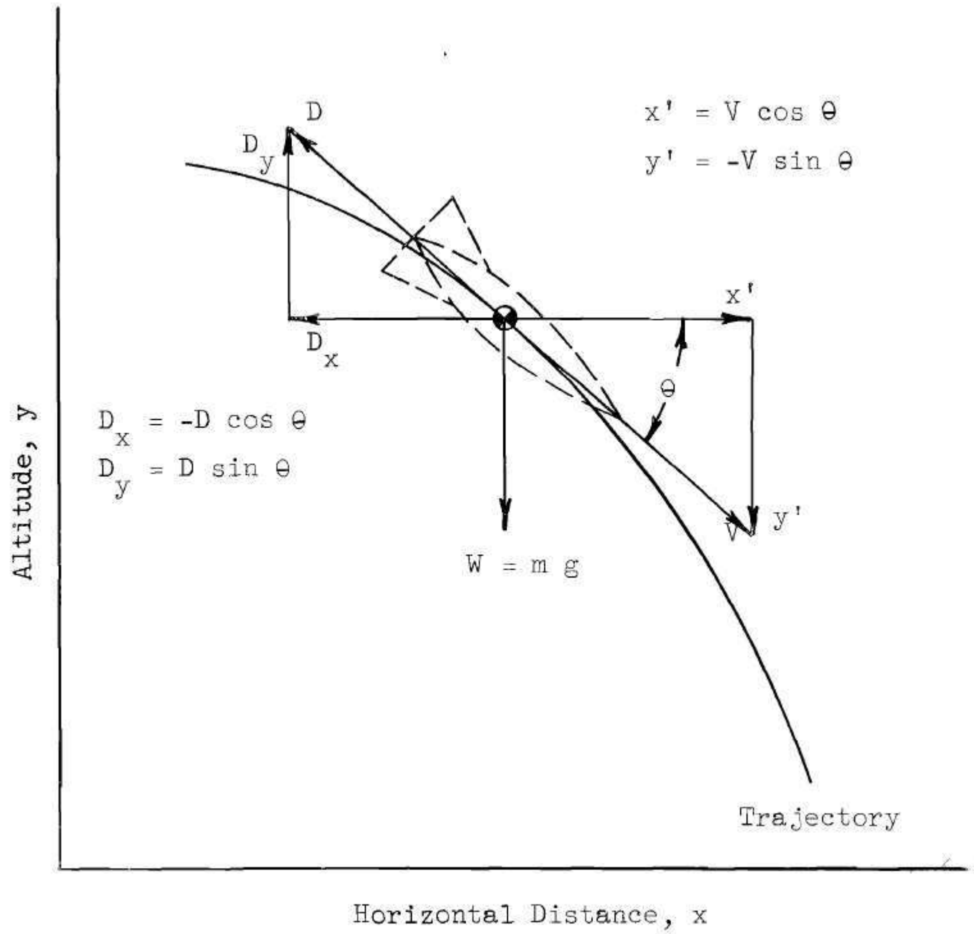

Figure 2. Sketch of Dynamics of a Missile During a Trajectory

and
$$y'' = (D/m) \sin \Theta - g. \tag{5}$$

The drag force may be represented by its equivalent relation from basic aerodynamics as:

$$D = 1/2\ \rho V^2\ S\ C_D \tag{6}$$

where ρ is the local density of the atmosphere, V is the velocity of the missile, S is the frontal area of the missile, and C_D is the drag coefficient of the missile. C_D is based on the frontal area, S, of the missile where:

$$S = \pi\ d^2/4 \tag{7}$$

d being the diameter of the missile. The mass, m, of the missile may be represented as:

$$m = W/G. \tag{8}$$

Substitution of equations (6) and (8) into equations (4) and (5) yields the relations:

$$x'' = -\ 1/2\ g\ C_D\ (S/W)\ \rho V^2 \cos \Theta \tag{9}$$

and
$$x'' = 1/2\ g\ C_D\ (S/W)\ \rho V^2 \sin \Theta - g. \tag{10}$$

The horizontal and vertical components of the velocity may be written as:

$$x' = V \cos \Theta \tag{11}$$

and
$$y'' = -\ V \sin \Theta \tag{12}$$

whereby the velocity of the missile may be expressed as:

$$V = \left[(x')^2 + (y')^2\right]^{0.5}. \tag{13}$$

Use of equations (11), (12) and (13) enables equations (9) and (10) to be expanded to:

$$x'' = -1/2\ g\ C_D\ (S/W)\ \rho\ \left[(x')^2 + (y')^2\right]^{0.5} x' \tag{14}$$

$$y'' = -1/2\ g\ C_D\ (S/W)\ \rho\ \left[(x')^2 + (y')^2\right]^{0.5} y' - g. \tag{15}$$

By definition of a new variable, $\overline{C}_D$, as:

$$\overline{C}_D = C_D/C_{D_{min}} \tag{16}$$

and definition of a ballistic coefficient, K, as:

$$K = -1/2\ g\ C_{D_{min}} \left[1/(W/S)\right] \tag{17}$$

equations (14) and (15) may be further written as

$$x'' = K\ \overline{C}_D\ \rho \left[(x')^2 + (y')^2\right]^{0.5} x' \tag{18}$$

and

$$y'' = K\ \overline{C}_D\ \rho \left[(x')^2 + (y')^2\right]^{0.5} y' - g. \tag{19}$$

The ballistic coefficient, K, embodies the physical characteristics of the missile and introduces the important ratio of the missile weight to the missile frontal area. Equations (18) and (19) are considered to be the equations of motion for this study, and it is readily seen that the ballistic coefficient is a most important factor in the determination of the trajectories. Analytical solutions to equations (18) and (19) are not easily obtained since the equations are non-linear.

A special case results when the medium is considered to have zero density, that is the medium is a vacuum. For this special case, the medium will have no resistance effect upon the missile, and the drag coefficient of the missile will be equal to zero. The ballistic coefficient, K, will also be zero in a vacuum, as is seen by the introduction of a zero in equation (17) for the $C_{D_{min}}$. In other words, for K = 0, equations (18) and (19) become:

$$x'' = 0 \tag{20}$$

and

$$y'' = -g \tag{21}$$

Equations (20) and (21) are recognized as the equations of motion for an arbitrary mass in a vacuum, and solution of these linear equations is easily accomplished by double integration. In this manner, the x and y coordinates at any time, t, may be found as:

$$x = x_o + x_o' t \tag{22}$$

and

$$y = y_o + y_o' t - 1/2 \, g \, t^2. \tag{23}$$

For the purposes of this study, the initial horizontal velocity, x_o', and the initial altitude, y_o, were considered to comprise the initial conditions of the trajectories. The initial horizontal displacement, x_o, and the initial vertical velocity, y_o', were chosen to be zero, such that equations (22) and (23) may be simplified to:

$$x = x_o' t \tag{24}$$

and

$$y = y_o - 1/2 \, g \, t^2. \tag{25}$$

CHAPTER III

METHODS OF SOLUTION

Since an adequate closed-form solution to equations (18) and (19) could not be established, approximate methods were next considered. Two different methods for solving the equations were attempted, these being the solution by analog computer methods and the solution by the method of Runge and Kutta utilizing digital computer techniques.

Analog computer.--Solution of differential equations by electrical analog methods is based on the analogy between the variation of voltages and the numerical variation of the desired functions and their derivatives.

The Analog Computer Laboratory facilities of the Georgia Institute of Technology were utilized for an initial exploration in the study of trajectories. Trajectories were obtained directly in graphical form by analog solution of the equations of motion. For this initial study, a range of ballistic coefficients between $K = 0$ and $K = -0.4$ was considered at each selected initial altitude and for various initial horizontal velocities. The initial altitudes chosen were 30, 50, 70 and 100 thousand feet, while the initial horizontal velocities were considered as 300, 600, 900 and 1200 feet per second. Thus, sixteen families of trajectories were obtained for the initial exploration. (A family of trajectories is defined for the purposes of this study as being those several trajectories obtained for a missile with varied ballistic coefficients but otherwise having identical initial altitude and initial horizontal velocity).

<u>IBM 650 Digital Computer (method of Runge and Kutta)</u>.--Solution of differential equations by the method of Runge and Kutta utilizes the principle of numerical integration and a description of the method may be found in Reference 1. The numerical feature of the method of Runge and Kutta causes it to be particularly amenable to digital computer machinery.

To verify the results of the analog method of solution, families of trajectories for the same initial conditions as were selected for the analog solution were also obtained by the method of Runge and Kutta as computed on the IBM 650 Digital Computer of the Georgia Institute of Technology. The range of ballistic coefficients selected for solution by means of the IBM 650 varied between $K = 0$ and $K = -0.003$. The Bell General Purpose System for programming the problem was selected because of its simplicity. The Fortran method of programming was also attempted but the lack of numerical storage cells within the computer caused the program obtained by the Fortran method to be invalid. The slow rate of output of the data by the Bell General Purpose System program as well as the lack of sufficient numerical storage cells within the computer indicated the need for a faster and larger computer.

<u>IBM 704 Digital Computer (method of Runge and Kutta)</u>.-- The Sandia Corporation, which sponsored this study, provided an IBM 704 Digital Computer as the means for obtaining the final data needed for this study. This type of computer possesses the desired features of large

numerical storage capacity and computational rapidity. Families of trajectories, and the location of the convergence point to which the constant time lines were found to converge, were obtained with the use of this computer for each of the initial conditions as are shown in Table 1. The ballistic coefficients considered for the final data have the corresponding W/S ratios as are shown in Table 2.

Table 1. Initial Conditions of Trajectories

Initial Altitude (feet)	Initial Horizontal Velocity (feet per second)	Ballistic Coefficient K
10,000	300	0.00
20,000	400	-0.02
30,000	500	-0.04
40,000	600	-0.06
50,000	700	-0.08
60,000	800	-0.10
80,000	900	
100,000	1000	
	1100	
	1200	

(Coordinate data were provided at every two second interval of the trajectory, terminating with the last two second interval having a non-negative altitude).

Table 2. Ratios of Missile Weight to Frontal Area Considered

Weight to Frontal Area Ratio W/S	Corresponding Ballistic Coefficient K
Infinity (Vacuum)	0.00
461.3	-0.02
230.6	-0.04
153.7	-0.06
115.3	-0.08
92.3	-0.10

For the missile considered in this investigation:

$$K = -9.2253/(W/S)$$

CHAPTER IV

DISCUSSION OF RESULTS

Analog computer.--The analog method of solution of the equations of motion was found to provide a rapid visual means for study of the computed trajectories. Each family of trajectories, as it was computed, was graphically plotted by the output equipment of the analog computer.

The data obtained by this method consisted of a dot placed on a sheet of graph paper at every five second interval along each trajectory, the dot having a known pair of coordinates with respect to the coordinate system selected. After a complete family of trajectories was obtained in this manner, the sheet of graph paper was then analyzed as to the mutual relationship of the dots or time-space coordinates. Each group of dots which represented the coordinates of the trajectories at equal time, t, were noted to lie in essentially a straight line for those trajectories having ballistic coefficients in the range between $K = 0$ and $K = -0.15$. The trajectories having ballistic coefficients more negative than $K = -0.15$ were found to deviate from the straight line relationship and were neglected. Justification for neglecting these trajectories was based on the fact that the W/S ratios for these deviating trajectories were of the order of magnitude of 50 pounds per square foot or less and are considered to be physically improbable.

Extension of the linear constant time lines toward the origin of the trajectories produced the phenomenon of the constant time lines converging to an extremely small circle of mutual intersection. For all practical purposes, the circle was of small enough radius to be considered as a point. Figure 3 is an illustration of the phenomenon, which had been noted previously in preliminary studies conducted at the Sandia Corporation.

For each of the sixteen families of trajectories computed by the analog method, the constant time lines were found to be essentially linear and were found to converge to a point. The location of the convergence point was found to vary horizontally with the initial conditions of each family of trajectories but did not appear to vary in a vertical direction from the initial altitude. In addition, the horizontal location of the convergence point was found to have a positive coordinate for those families having an initial horizontal velocity of 300 feet per second and having initial altitudes of 50, 70 and 100 thousand feet. Location of the convergence point for these families of trajectories, as characterized by Figure 4, indicated that time lines constructed between the convergence point and the computed points of the earlier portion of the vacuum trajectory did not have the proper orientation with respect to the expected progression of the coordinates of the other trajectories (those having non-zero ballisitc coefficients).

The adverse effect of a positive $\overline{x}$ does not cause the hypothesis to be abandoned, however, since an additional resolving effect was also noted to occur in the families of trajectories having the positive $\overline{x}$. This resolving effect indicated that the portions of the trajectories which lie to the more negative side of the convergence point may be very closely approximated by the vacuum trajectory. Explanation of this

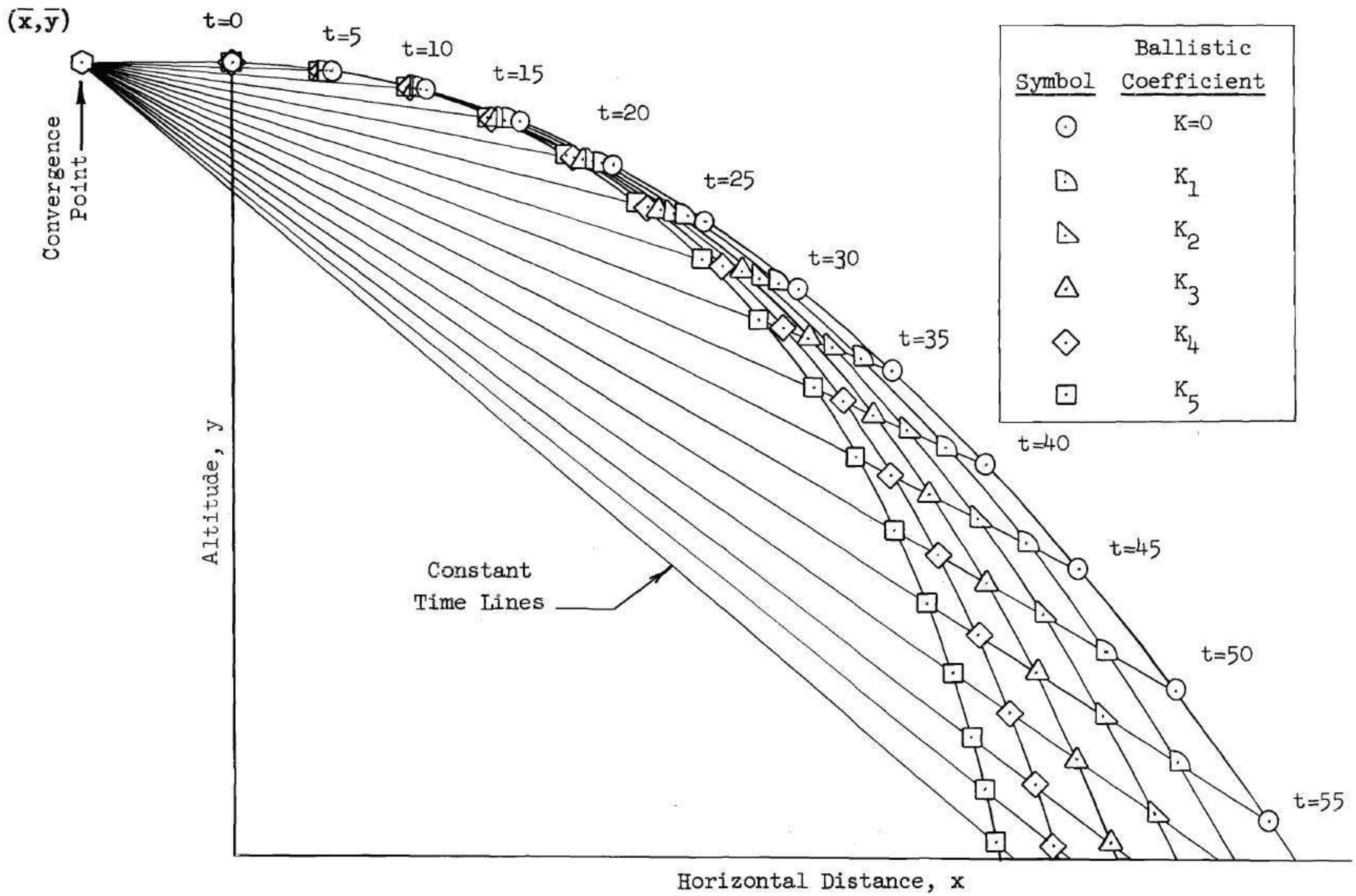

Figure 3. Sketch of a Family of Trajectories with Constant Time Lines

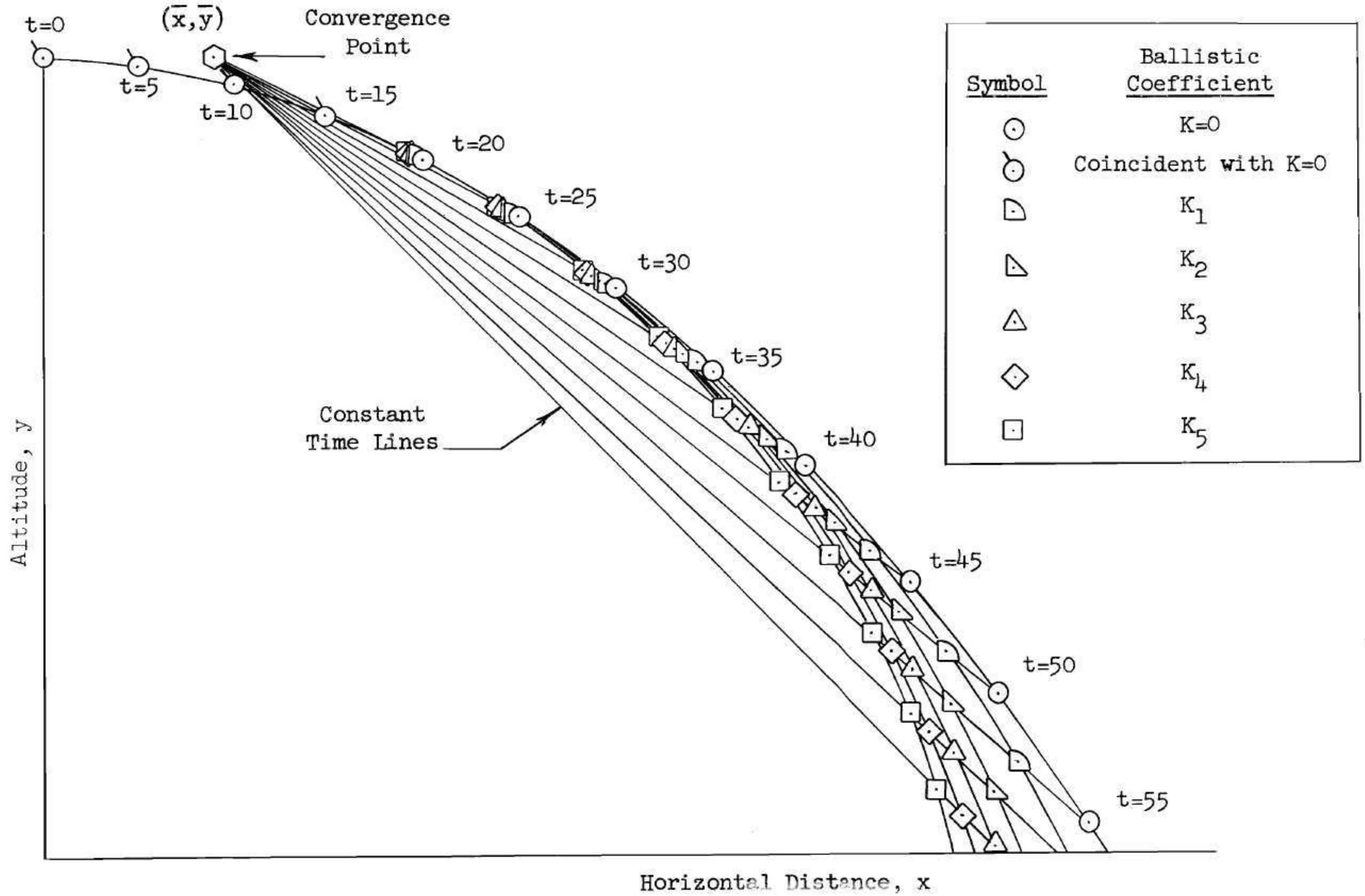

Figure 4. Sketch of a Family of Trajectories With a Positive $\bar{x}$

resolving effect appears to be that, for the families of trajectories noted as having this feature, the gravity force is greatly predominant over the drag force during the early portions of the trajectories.

A severe restriction to the usefulness of the analog data is that the small errors inherent to analog equipment appeared to be magnified greatly by the scale factors necessarily applied during the computations. Scaling-down of the variables so as to be represented by attainable voltages is necessary during the input phases of the computation, but scaling-up of end results causes electrostatic noise within the scaling-up machinery to introduce appreciable error into the results. Accuracy of the analog computed results was therefore somewhat questionable.

IBM 650 Digital Computer.--The trajectories obtained by use of the IBM 650 Digital Computer were for the very small range of ballistic coefficients between K = 0 and K = -0.003. These data were found to agree, in general, quite well with the analog computed data, the accuracy to which the graphical results of the analog data could be interpreted (approximately to the nearest 250 feet) accounting for the disagreement between comparable data.

Numerically constructed time lines for this data further supported the existence of the convergence phenomenon of linear constant time lines and further indicated that the location of the convergence point was a definite function of the initial conditions. An additional observation from this data, as illustrated in Figure 5, was that the spacing of the coordinates of the trajectories varied along constant time lines practically linearly as the ballistic coefficient varied.

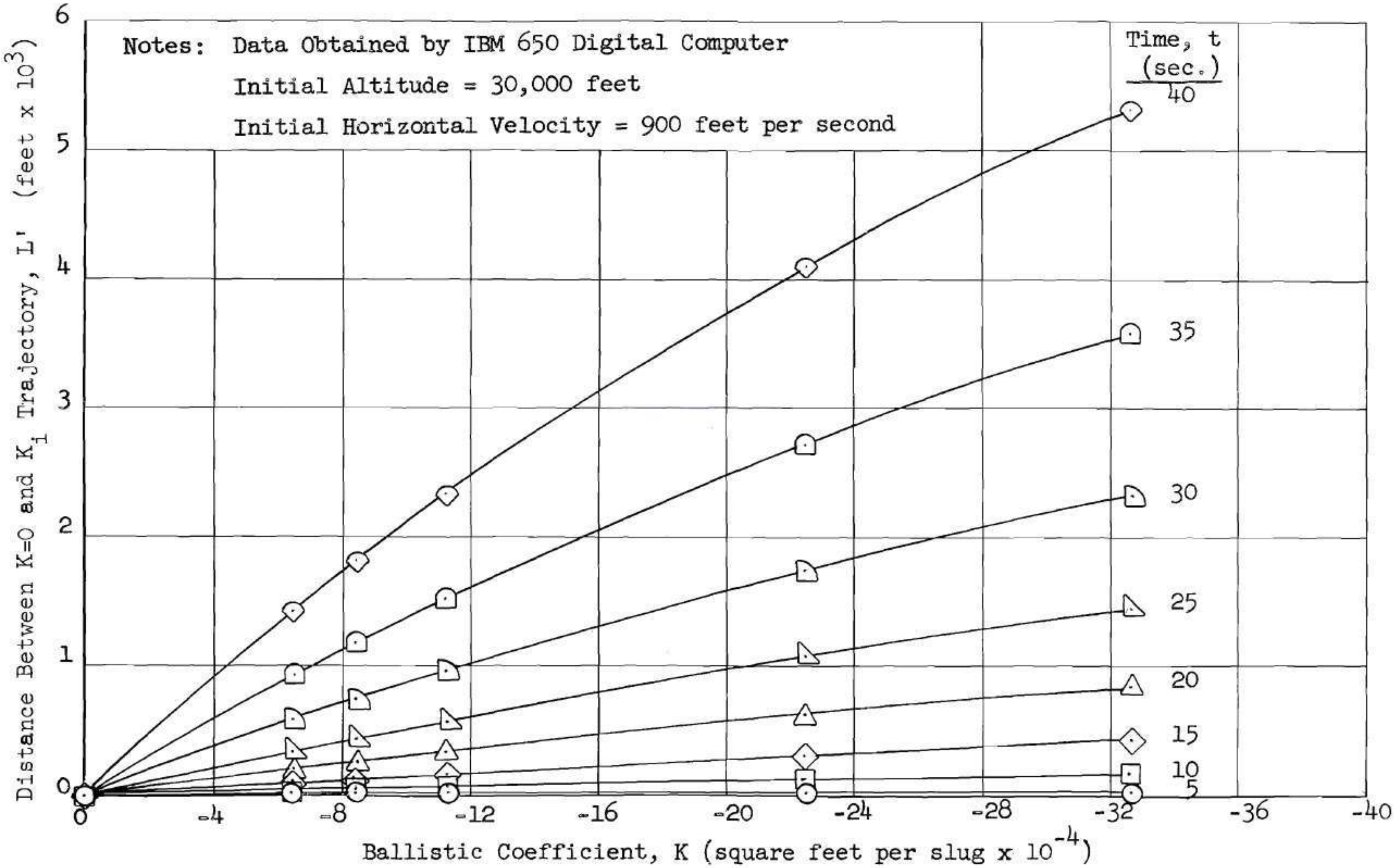

Figure 5. Sample Variation of Trajectories Along Constant Time Lines

This fact indicates that, for small changes in ballistic coefficient, linear interpolation between two known trajectories along constant time lines will provide a means for predicting intermediate trajectories.

The prohibitive amount of time needed for the computation of the trajectories by the Bell General Purpose System prompted investigation of solution by faster and more flexible digital methods.

IBM 704 Digital Computer.--The IBM 704 Digital Computer of the Sandia Corporation was obtained as the means by which the data necessary for investigating the large number of parameters of this study were to be computed. This type of digital computer is exceedingly fast and has a very large numerical storage capacity. Trajectory data were provided by this computer for each of the initial conditions as are shown in Table 1, the data being computed and stored in such a way as to enable the computer to numerically locate the optimum point to which the linear constant time lines of each family of trajectories would converge.

Location of the convergence point was achieved by an iterative process, using the criteria of minimizing the maximum relative error. (The relative error for the purposes of this study was considered to be the ratio of the distance by which the numerically constructed time line deviated from the computed trajectory point to the altitude of the computed point). A first approximation to the location of the convergence point was obtained by numerical construction of time lines so as to fit the group of points comprising a constant time group in the best least square sense. The center of the small circle of mutual intersection of the constructed time lines was taken to be the first approximation to the point. Translation of the point from its first approximation location to other nearby locations will cause the slope of the time lines, as well as the relative errors, to change accordingly. Iteration upon the location of the point may be terminated when the maximum relative error throughout the field of the trajectory data cannot be appreciably reduced by further iteration. The horizontal location and vertical deviation of the point from the initial altitude, as found by the described iteration method, are graphically presented in Figures 6 and 7 respectively. A tabulation of the corresponding maximum relative errors and the altitude at which the maximum relative errors occurred may be found in Table 3.

The location of the maximum relative error within each family of trajectories generally occurs at the lower time lines and generally occurs at the trajectory which is furthest from the vacuum trajectory. An explanation for the maximum relative error generally occurring at the lower time lines lies in the fact that, at these lines, the altitudes are reasonable small. Therefore, the ratio of the absolute error (in feet) to the altitude may be relatively large even though the absolute error in itself may be quite small. The errors thus obtained are felt to be not excessive when consideration is made that a prediction of a trajectory with small errors is very valuable in situations where large scale computers are not readily available to compute the actual trajectory.

The location of the convergence point as a function of initial altitude and initial horizontal velocity is as shown in Figures 6 and 7 for the particular missile studied. The variation of the convergence point horizontally and deviation of the convergence point vertically

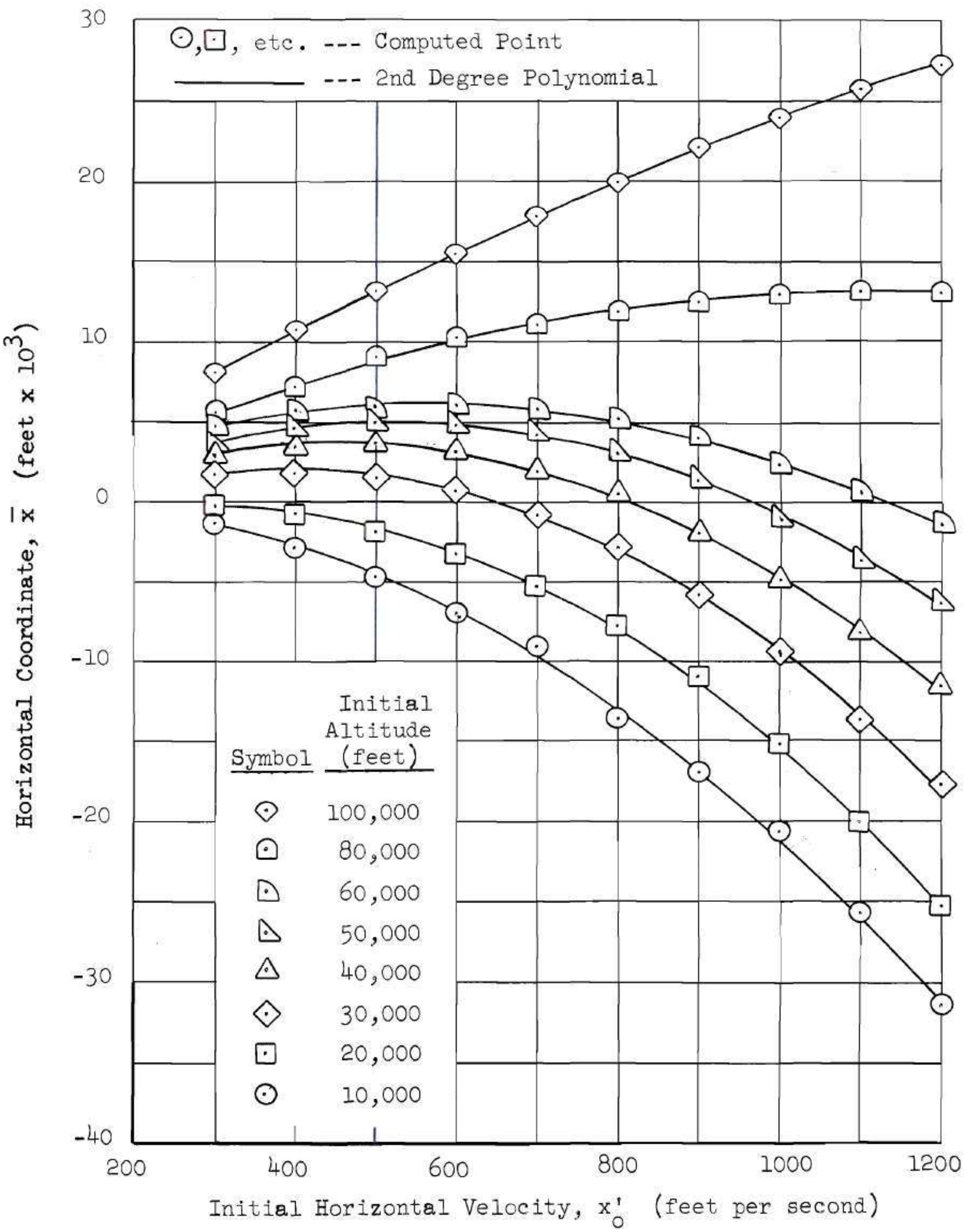

Figure 6. Horizontal Coordinate of Convergence Point

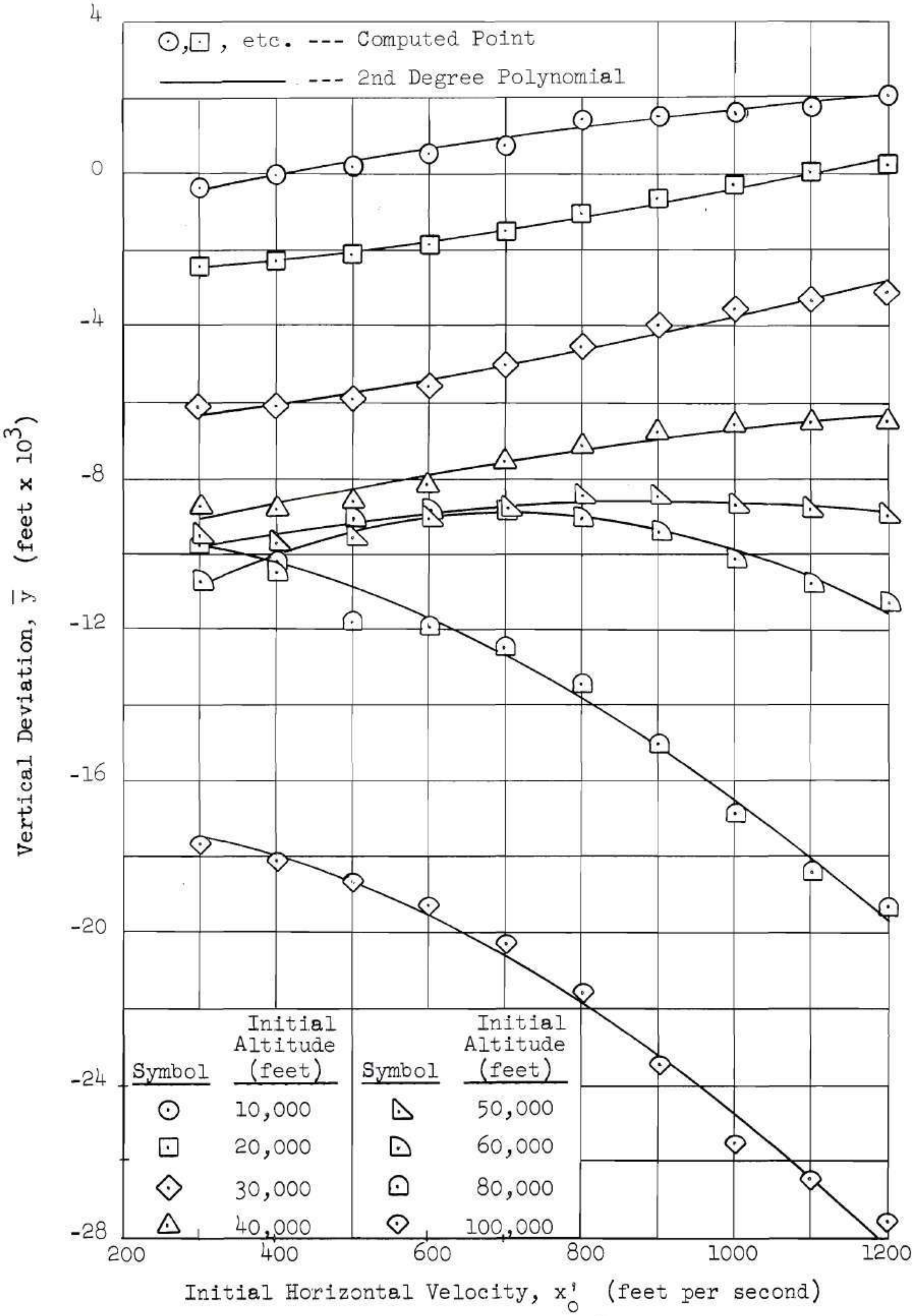

Figure 7. Vertical Deviation of Convergence Point

Table 3. Maximum Relative Errors Obtained (Per Cent)

Initial Velocity (fps)	Initial Altitude (thousands of feet)							
	10	20	30	40	50	60	80	100
300	0.718 (1880)	2.056 (17368)	4.566 (24819)	4.760 (33509)	3.129 (41955)	2.142 (50247)	1.576 (23647)	------ ------
400	0.810 (1986)	2.294 (16751)	4.975 (24928)	5.276 (33622)	3.509 (42064)	2.798 (18207)	2.190 (23929)	1.750 (28042)
500	0.926 (2094)	2.356 (16827)	5.077 (25861)	5.430 (33733)	4.057 (15245)	3.696 (18635)	2.804 (24277)	2.262 (28295)
600	1.054 (2201)	2.332 (16192)	4.939 (25126)	5.224 (33839)	4.634 (15901)	4.304 (19133)	3.439 (24684)	2.811 (28594)
700	1.159 (2302)	2.189 (16264)	4.485 (25212)	4.821 (34868)	5.047 (16422)	4.893 (19682)	4.122 (25137)	3.428 (28932)
800	0.785 (2405)	1.919 (16325)	4.073 (25286)	4.594 (34028)	5.212 (16960)	5.277 (20256)	4.804 (25618)	4.122 (29299)
900	0.647 (9200)	1.679 (15657)	3.747 (25348)	4.489 (34109)	4.911 (17469)	5.359 (20810)	5.494 (26099)	4.934 (29680)
1000	1.251 (4596)	1.644 (7479)	3.534 (24572)	4.506 (34172)	4.264 (43807)	5.256 (21304)	6.306 (26546)	5.876 (30057)
1100	3.006 (2721)	2.974 (6101)	3.420 (23753)	4.593 (34226)	4.598 (43822)	5.443 (21760)	7.286 (26959)	6.968 (82173)
1200	4.520 (2796)	4.412 (6235)	3.399 (23790)	4.700 (33333)	4.887 (43953)	5.921 (22188)	8.180 (27347)	7.872 (84090)

The number within the parentheses beneath the maximum relative error indicates the altitude at which the maximum relative error was obtained.

from the initial altitude are designated as $\overline{x}$ and $\overline{y}$ respectively, and are considered in reference to the initial point of the trajectory (i. e. $t = 0$). These variations in the point location were found to be adequately described, as shown in Figures 6 and 7, by a polynomial of the second degree whose coefficients vary with the initial altitude but not with the initial horizontal velocity. These polynomials were evaluated by the method of least squares, and a description of this method may be found in Reference 3. The polynomials provide a simple means for predicting $\overline{x}$ and $\overline{y}$ for any desired initial horizontal velocity at any of the initial altitudes for which the coefficients of the polynomial have been evaluated. The coefficients of the second degree polynomials may be found in Table 4. Inspection of Table 4 indicates that the coefficients are no simple function of initial altitude; therefore, linear interpolation to obtain the coefficients for intermediate altitudes is not recommended.

To provide yet another means for predicting $\overline{x}$ and $\overline{y}$, nomographs were constructed by the methods described in Reference 4. These nomographs, Figures 8 and 9 respectively, enable the $\overline{x}$ and $\overline{y}$ corresponding to any combination of initial altitude and initial horizontal velocity to be found quickly without any computation whatsoever. Figure 9 is found to be quite accurate throughout the entire range of the independent variables studies. Figure 8, on the other hand, may give errors of the order of 2000 feet for the combinations of the lower velocities (below 700 feet per second) with the higher initial altitudes (above 50,000 feet). Further refinements to the nomograph construction methods appear possible and these refinements should produce more accurate nomographs.

Table 4. Coefficients of Approximating Second Degree Polynomial

Horizontal Coordinate of Convergence Point, $\overline{x}$ $(\overline{x} = p_1 + p_2 V + p_3 V^2)$

Altitude (feet)	p_1	p_2	p_3
10,000	-7.582×10^2	5.089	-2.532×10^{-2}
20,000	-32.789×10^2	19.047	-3.108×10^{-2}
30,000	-29.789×10^2	24.803	-3.106×10^{-2}
40,000	-21.688×10^2	25.606	-2.796×10^{-2}
50,000	-19.752×10^2	26.554	-2.531×10^{-2}
60,000	-1.396×10^2	21.944	-1.928×10^{-2}
80,000	-9.368×10^2	25.073	-1.118×10^{-2}
100,000	-3.467×10^2	30.119	-0.576×10^{-2}

Vertical Deviation of Convergence Point, $\overline{y}$ $(\overline{y} = p_4 + p_5 V + p_6 V^2)$

Altitude (feet)	p_4	p_5	p_6
10,000	-16.820×10^2	4.502	-0.112×10^{-2}
20,000	-31.083×10^2	1.523	0.120×10^{-2}
30,000	-72.370×10^2	2.669	0.082×10^{-2}
40,000	-104.686×10^2	5.030	-0.130×10^{-2}
50,000	-113.926×10^2	5.977	-0.325×10^{-2}
60,000	-145.229×10^2	15.678	-1.103×10^{-2}
80,000	-90.488×10^2	-0.520	-0.694×10^{-2}
100,000	-168.974×10^2	0.592	-0.843×10^{-2}

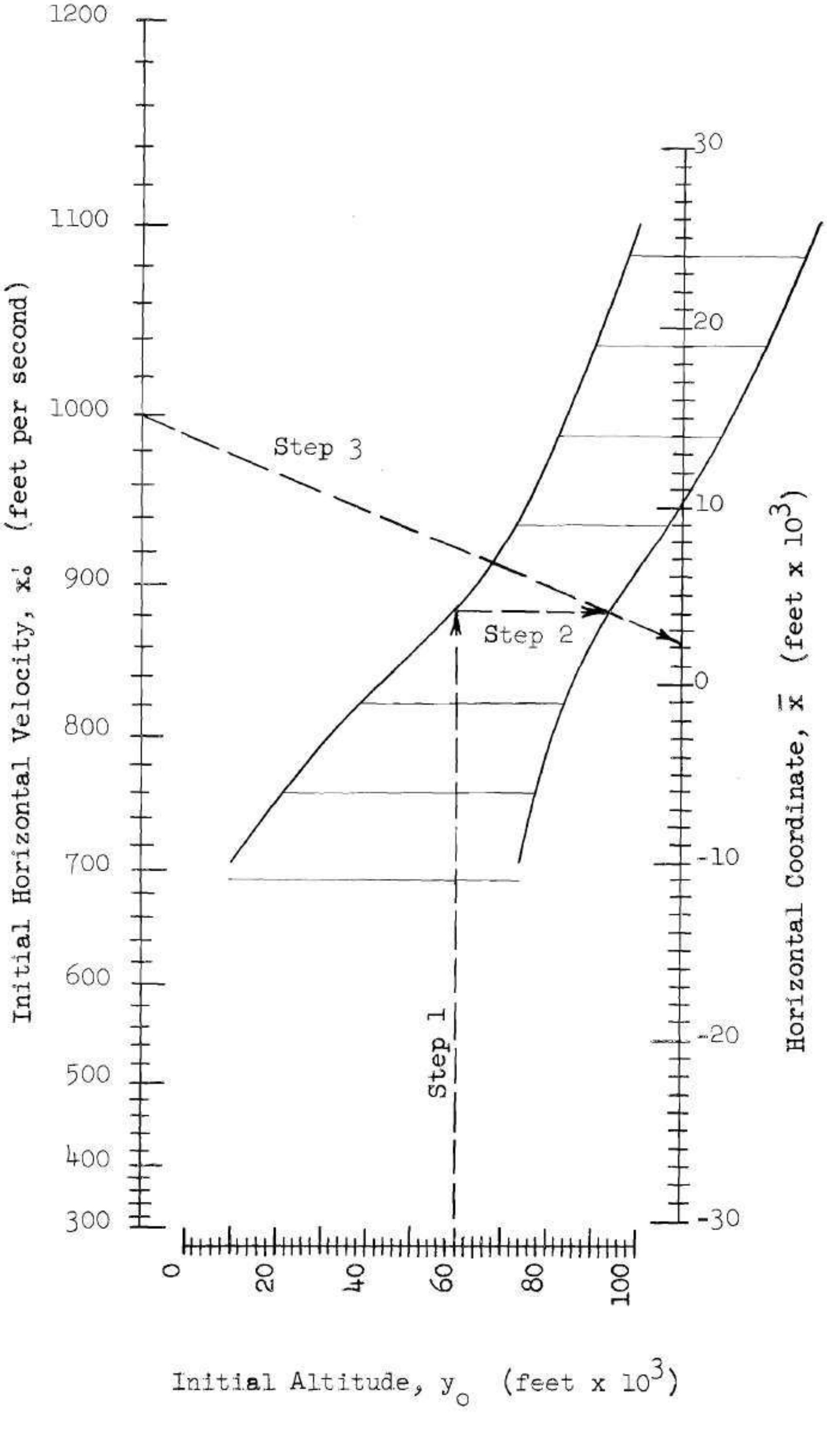

Figure 8. Nomograph for Predicting Horizontal Coordinate of the Point

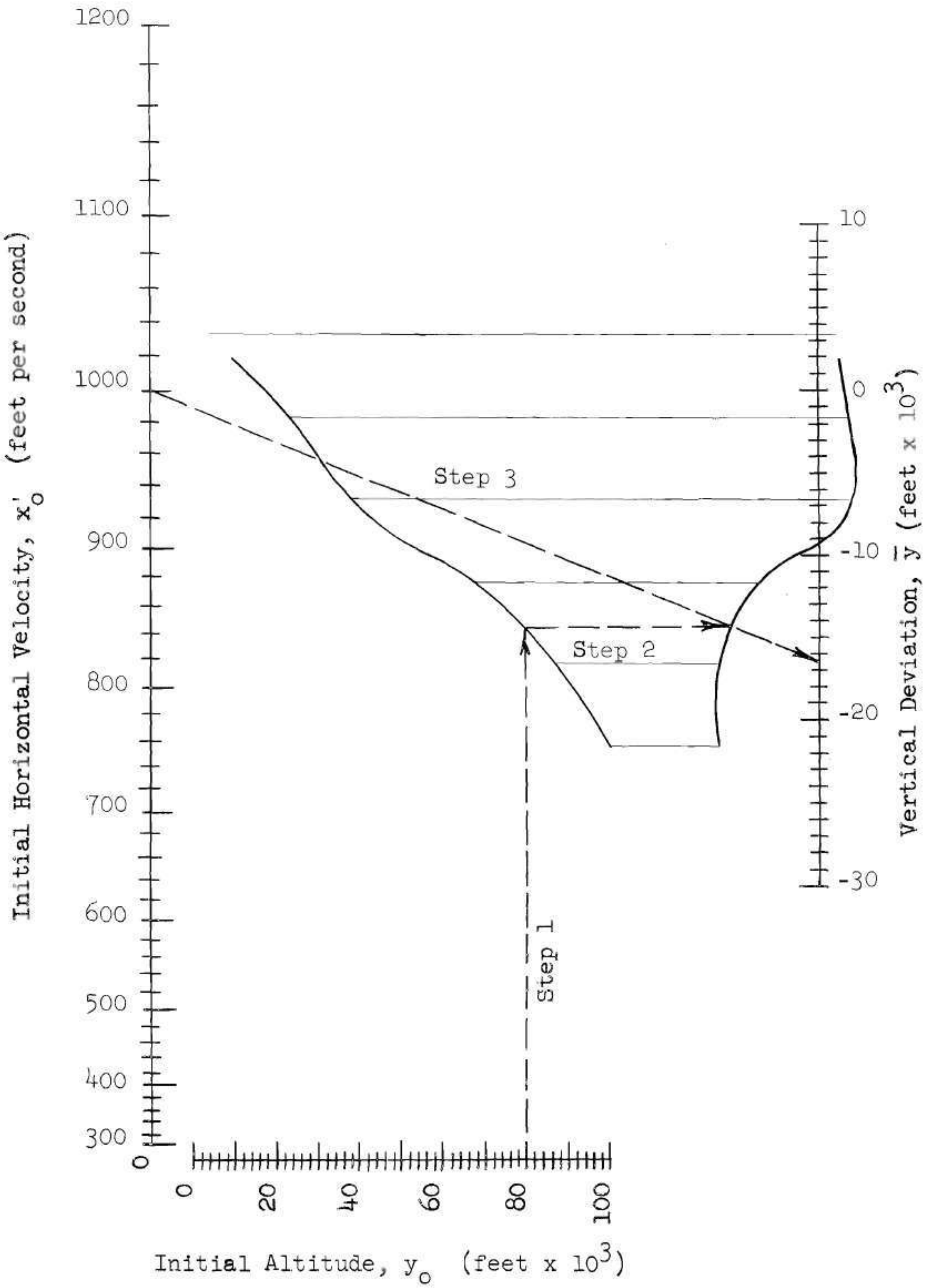

Figure 9. Nomograph for Predicting Vertical Deviation of the Point

CHAPTER V

CONCLUSIONS

Within the limitations of the original assumptions needed for the derivation of the equations of motion, the accuracy of the method of solution of the equations of motion, and the range of initial altitudes, initial horizontal velocities and ballistic coefficients considered, it is concluded for the missile studied that:

1. The constant time lines of a family of trajectories may be considered to be linear and to converge to a point.

2. A point exists for each family of trajectories from which constant time lines may be constructed, in conjunction with a vacuum trajectory, such that the maximum relative error throughout the field of the trajectory data will be a minimum.

3. Nomographical results presented, by which the location of the point of convergence may be described as a function of the initial conditions, require further refinement.

4. A polynomial of the second degree adequately describes the location of the convergence point as a function of the initial horizontal velocity at a constant initial altitude.

5. Only physically probable W/S ratios in conjunction with the applicable vacuum trajectory should be considered in determining the location of the convergence point for each family of trajectories.

6. Linear interpolation along constant time lines may be used to predict trajectories provided the range of interpolation is small.

CHAPTER VI

RECOMMENDATIONS

It is recommended that:

1. A study be made to determine if other types of missiles possess the property of essentially linear time lines and have a point of convergence of constant time lines for each family of trajectories.

2. Investigation be conducted to determine the manner with which the non-zero ballistic coefficient trajectories deviate from the vacuum trajectory along constant time lines.

3. A study be made of the maximum relative errors incurred by restricting the vertical location of the convergence point to be at the initial altitude, thus allowing the convergence point to have only horizontal variation with the initial conditions.

4. Further attempts be made to refine the nomographical method of presentation of the location of the convergence point as a function of the initial condition of the trajectories.

5. A study be conducted of the trajectories obtained by solution of the equations of motion which have an initial vertical velocity component included in the initial conditions.

6. The data obtained by the methods of this sutdy be verified by comparison with available experimental data.

APPENDIX

Relations for Density and Speed of Sound

(Summarized from Reference 2)

Altitudes Up To 35,332 Feet:

Density $$\rho = 0.002378 \left[(145{,}366 - y)/145{,}366 \right]^{4.255}$$

Speed of Sound $$c = 1117.0 \left[(145{,}366 - y)/145{,}366 \right]^{0.5}$$

Altitudes Between 35,332 Feet and 104,987 Feet:

Density $$\rho = 0.00314158 \left[10^{((4705 - y)/48{,}211)} \right]$$

Speed of Sound $$c = 971.0$$

Density has the units of slugs per cubic feet. Speed of sound has the units of feet per second.

Empirical Relations Defining $\overline{C}_D$ Variation with Mach Number

(Used in IBM 650 Digital Computer Method)

Mach numbers less than 0.60:

$$\overline{C}_D = 1.000$$

Mach numbers between 0.60 and 1.27:

$$\overline{C}_D = 2.139 + 0.89049 \arctan \left[9.0 \ (M - 0.97) \right]$$

Mach numbers above 1.27:

$$\overline{C}_D = 3.250 + 0.98113 \ (M - 1.27)$$

where $\overline{C}_D = C_D \ / \ C_{D_{min}}$ and $C_{D_{min}} = 0.573$

REFERENCES

1. Kooy, J. M. J., and Uytenbogaart, J. W. H., Ballistics of the Future, Technical Publishing Company, Haarlem, Holland, 1947.

2. Staff of Ames Aeronautical Laboratory, Notes and Tables for Use in the Analysis of Supersonic Flow, NACA Technical Note 1428, Washington, D. C., 1947.

3. Milne, W. E., Numerical Calculus, Princeton University Press, Princeton, New Jersey, 1949.

4. Burrows, W. H., Direct Construction of Nomographs from Tables, Industrial and Engineering Chemistry, Volume 47, 1955.